Sum Rules of the Multiple Giant Dipole States

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Various sum rules for multiple giant dipole resonance states are derived. For the triple giant dipole resonance states, the energy-weighted sum of the transition strengths requires a model to be related to those of the single and double giant dipole resonance states. It is also shown that the non-diagonal matrix elements of the double commutator between the dipole operator and the nuclear Hamiltonian give useful identities for the excitation energy and transition strength of each excited state. Using those identities, the relationship between width of the single dipole state and those of the multiple ones is qualitatively discussed.

It is well known that sum rules play an important role in a wide range of physics.[1] On the one hand, if an experiment shows breaking of the appropriate sum rule, the basic framework of the model should be improved. For example, the enhancement of the Thomas-Reiche-Kuhn(TRK) sum rule value requires meson-exchange currents.[1,2] The quenching of the Coulomb[3] and Ikeda[4] sum rule values indicates that nucleon-degrees of freedom are not enough for understanding of the relevant nuclear phenomena. On the other hand, when we make some approximations in calculations, we should take care of the fundamental sum rules. From this point of view, the sum rules in the random phase approximation have been explored by many authors in non-relativistic[5] and relativistic[6] frameworks.

In nuclear physics, a typical example of the sum rules is obtained by the relationship

$$[D, [H, D]] = C, \tag{1}$$

where H, D, and C denote the Hamiltonian of the system, the excitation operator, and a c-number, respectively. When we take the ground-state expectation value of the both sides, we have the model-independent sum rule,

$$2 \sum_n \omega_n |\langle n \,|\, D \,|\, \rangle|^2 = C, \tag{2}$$

where ω_n represents the excitation energy of the eigen state of the Hamiltonian $|n\rangle$ from the ground state $|\,\rangle$. For example, the TRK sum rule for dipole excitations,

$$\sum_n \omega_n |\langle n \,|\, D \,|\, \rangle|^2 = \frac{1}{2m} \frac{NZ}{A}, \tag{3}$$

is obtained for

$$D = \frac{Z}{A}\sum_{i=1}^{N} z_i - \frac{N}{A}\sum_{i=1}^{Z} z_i, \tag{4}$$

assuming its double commutator with the nuclear interactions to be zero. Recently in the same approximation as for the TRK sum rule, another sum rule has been derived,[7]

$$\sum_n \omega_n |\langle n \,|\, D^2 \,|\, \rangle|^2 = 4S_0(1)S_1(1), \tag{5}$$

where $S_1(1)$ stands for the TRK sum rule value and $S_0(1)$ is the non-energy weighted sum of the dipole transition strengths,

$$S_0(1) = \sum_n |\langle n \,|\, D \,|\, \rangle|^2. \tag{6}$$

The new sum rule Eq.(5) is related to the double giant dipole resonance(DGDR) states which have been recently observed through relativistic heavy ion reactions.[8] It implies that the energy-weighted sum of the transition strengths for the DGDR is determined by the excitation energies and strengths of the single giant dipole resonance(SGDR) states. This relationship is useful both for the analysis of experimental data and for making models of the DGDR, since the SGDR is well known.

The purpose of the present paper is to derive other sum rules for multiple dipole states on the basis of the same assumption as for the TRK sum rule. In particular, we will discuss the sum rules for the operator D^3, which is related to the triple giant dipole resonance(TGDR) states. We will also show that the non-diagonal matrix elements of Eq.(1) provide us with useful identities to understand the structure of the multiple giant resonance states. They give a constraint not only on the sum of the matrix elements, but also on each energy-weighted matrix element.

First we discuss sum rules for the operator D^3. For convenience, let us define the notation for the sum of the matrix elements,

$$S_k(m) = \sum_n \omega_{mn}^k |\langle mn \,|\, D^m \,|\, \rangle|^2, \tag{7}$$

where ω_{mn} denotes the excitation energy of the n-th state of the m-times SGDR, $|mn\rangle$. In this notation, the TRK sum rule Eq.(3) and the new sum rule Eq.(5) mentioned above are expressed as

$$S_1(1) = \sum_n \omega_{1n} |\langle 1n \,|\, D \,|\, \rangle|^2, \tag{8}$$

$$S_1(2) = \sum_n \omega_{2n} |\langle 2n \,|\, D^2 \,|\, \rangle|^2. \tag{9}$$

The above two sum rules are calculated with the use of the closure property of the intermediate states. For example, $S_1(2)$ is expressed as

$$S_1(2) = \frac{1}{2}\left(\sum_n \langle \,|\, [D^2, H] \,|\, 2n \rangle\langle 2n \,|\, D^2 \,|\, \rangle + \langle \,|\, D^2 \,|\, 2n \rangle\langle 2n \,|\, [H, D^2] \,|\, \rangle \right). \tag{10}$$

This equation is rewritten with the closure property as

$$S_1(2) = \frac{1}{2}\langle\,|[D^2,[H,D^2]]|\,\rangle. \tag{11}$$

By calculating explicitly the double commutator, we obtain Eq.(5).

Unfortunately, for $S_1(3)$ we can not use the same procedure, because the final states excited by D^3 are not only $|3n\rangle$, but also $|1n\rangle$. In the case of Eq.(10), D^2 has also a matrix element between the ground states, but it does not contribute to the energy-weighted sum and therefore, we could use the closure property. In other words, the sum rule value obtained by the double commutator of D^3 with Hamiltonian is not the value of $S_1(3)$. Keeping this fact in mind, we calculate

$$\begin{aligned}
S_1(f) &= \sum_f \omega_f |\langle\,f\,|D^3|\,\rangle|^2 \\
&= \frac{1}{2}\langle\,|[D^3,[H,D^3]]|\,\rangle = 9S_1(1)\left(S_0^2(1) + S_0(2)\right),
\end{aligned} \tag{12}$$

where we have used

$$\langle\,|D^4|\,\rangle = S_0^2(1) + S_0(2). \tag{13}$$

Thus the value of $S_1(f)$ is fixed by the TRK sum rule values and the transition strengths of the SGDR and DGDR. In the above equation, however, $|f\rangle$ should be either $|1n\rangle$ or $|3n\rangle$. Fig. 1 shows the transitions to be included in $S_1(f)$. In addition to (a) corresponding to $S_1(3)$, four kinds of transitions contribute to $S_1(f)$.
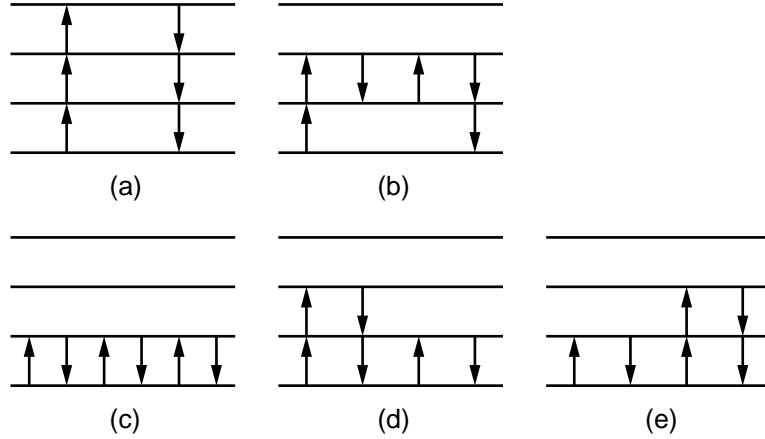

Fig. 1. Transitions induced by D^3

Among the transitions in Fig. 1, the sums of the strengths in (c), (d), and (e) are calculated individually. The contribution (c) is written as

$$\begin{aligned}
(c) &= \sum_{n',n'',n'''} \omega_{1n'}\langle\,|D|1n\rangle\langle1n|D|\,\rangle\langle\,|D|1n'\rangle\langle1n'|D|\,\rangle\langle\,|D|1n''\rangle\langle n''|D|\,\rangle \\
&= S_1(1)S_0^2(1).
\end{aligned} \tag{14}$$

The sums of (d) and (e), on the other hand, are described as

$$\text{(d)} = \text{(e)} = \sum_{n,n',n'',n'''} \omega_{1n''} \langle \,|D|\,1n\,\rangle\langle\,1n\,|D|\,2n'\,\rangle\langle\,2n'\,|D|\,1n''\,\rangle$$

$$\times \langle\,1n''\,|D|\,\rangle\langle\,|D|\,1n'''\,\rangle\langle\,1n'''\,|D|\,\rangle$$

$$= \frac{1}{2}\langle\,|[D^3,[H,D]]|\,\rangle S_0(1) - S_1(1)S_0^2(1) = 2S_0^2(1)S_1(1). \qquad (15)$$

From Eqs.(12), (14) and (15), we have the sum of (a) and (b)

$$S_1(3) + \sum \omega_{1n''}\langle\,|D|\,1n\,\rangle\langle\,1n\,|D|\,2n'\,\rangle\langle\,2n'\,|D|\,1n''\,\rangle$$

$$\times \langle\,1n''\,|D|\,2n'''\,\rangle\langle\,2n'''\,|D|\,1n''''\,\rangle\langle\,1n''''\,|D|\,\rangle$$

$$= 4S_1(1)S_0^2(1) + 9S_1(1)S_0(2). \qquad (16)$$

Thus, although several sum rules are obtained for D^3, we can not obtain the relationship of $S_1(3)$ to the energy-weighted and non-energy-weighted sum for the strengths of the SGDR and DGDR. The second term of the left hand side in the above equation, coming from (b), can not be expressed in terms of $S_k(m)$.

In order to separate the contribution $S_1(3)$ from (b), we need an assumption. For example, if we assume the folding model where the DGDR states are composed of the dipole bosons $|1n\rangle$,

$$|2n\rangle = |1n',1n''\rangle \quad (n' \geq n''), \qquad (17)$$

then we have

$$\text{(b)} = 4S_1(1)S_0^2(1). \qquad (18)$$

Here, we have used the equation of the folding model,

$$\sum_{n''}\langle\,1n\,|D|\,2n''\,\rangle\langle\,2n''\,|D|\,1n'\,\rangle = \delta_{nn'}S_0(1) + \langle\,|D|\,1n'\,\rangle\langle\,1n\,|D|\,\rangle, \qquad (19)$$

which is obtained from the fact that

$$\langle\,1n\,|D|\,1n',1n''\,\rangle = \sqrt{2}\langle\,|D|\,1n\,\rangle\delta_{nn'}\delta_{n'n''} + \langle\,|D|\,1n''\,\rangle\delta_{nn'}(1-\delta_{n'n''})$$

$$+ \langle\,|D|\,1n'\,\rangle\delta_{nn''}(1-\delta_{n'n''}). \qquad (20)$$

Eqs.(16) and (18) yield

$$S_1(3) = 9S_1(1)S_0(2). \qquad (21)$$

We note that Passos et al.[9] have defined TGDR states which are excited by the operator

$$O^{(3)} = D^3 - D\frac{\langle\,|D^4|\,\rangle}{\langle\,|D^2|\,\rangle}. \qquad (22)$$

This operator is determined with the requirement that the doorway state of the TGDR is orthogonal to the one of the SGDR,

$$\sum_n \langle \, | D | 1n \rangle \langle 1n | O^{(3)} | \, \rangle = 0. \tag{23}$$

Their TGDR states defined in this way are different from the present $|3n\rangle$ which satisfies $\langle 1n|3n'\rangle = 0$.

The sum rules discussed so far are related to the diagonal matrix elements of Eq.(1). Next we will show that if we calculate its non-diagonal matrix elements, we can obtain other useful identities.

The equation

$$\langle \, m+1, n \, | \, [D, [H, D]] \, | \, m-1, n' \, \rangle = 0 \tag{24}$$

provides us with

$$(\omega_{m+,n} + \omega_{m-1,n'}) \sum_{n''} \langle m+1, n \, | \, D \, | \, mn'' \rangle \langle mn'' \, | \, D \, | \, m-1, n' \rangle$$

$$= 2 \sum_{n''} \omega_{mn''} \langle m+1, n \, | \, D \, | \, mn'' \rangle \langle mn'' \, | \, D \, | \, m-1, n' \rangle. \tag{25}$$

For m=1 and 2, for example, we have the relationships,

$$\omega_{2n} \langle 2n \, | \, D^2 \, | \, \rangle = 2 \sum_{n'} \omega_{1n'} \langle 2n \, | \, D \, | \, 1n' \rangle \langle 1n' \, | \, D \, | \, \rangle, \tag{26}$$

$$\omega_{3n} \langle 3n \, | \, D^3 \, | \, \rangle = 3 \sum_{n'} \omega_{1n'} \langle 3n \, | \, D^2 \, | \, 1n' \rangle \langle 1n' \, | \, D \, | \, \rangle. \tag{27}$$

In deriving the last equation, we have used Eq.(26), together with Eq.(25). According to these identities, it is possible to make the following comments.

First, the excitation energy ω_{2n} is determined by ω_{1n} and transition strengths. This fact is also true for other ω_{mn}. The sum rule Eq.(5) is the relationship between the sum of the energy-weighted strengths for the DGDR and those of the SGDR, while Eq.(26) provides the relationship of each energy-weighted strength of $|2n\rangle$ to ω_{1n} and transition strengths,

$$\omega_{2n} \langle \, | D^2 | 2n \rangle \langle 2n | D^2 | \, \rangle = 2 \sum_{n'} \omega_{1n'} \langle \, | D^2 | 2n \rangle \langle 2n | D | 1n' \rangle \langle 1n' | D | \, \rangle. \tag{28}$$

In using the above equation, the energy-weighted sum of the DGDR is expressed in the form:

$$S_1(2) = 2 \sum_{n,n'} \omega_{1n'} \langle \, | D^2 | 2n \rangle \langle 2n | D | 1n' \rangle \langle 1n' | D | \, \rangle. \tag{29}$$

Since the right-hand side of this equation is expressed as

$$\text{r.h.s.} = 2 \sum_n \omega_{1n} \langle \, | D^3 | 1n \rangle \langle 1n | D | \, \rangle - 2S_0(1)S_1(1), \tag{30}$$

we have another sum rule,

$$\sum_n \omega_{1n} \langle \ | D^3 | 1n \rangle \langle 1n | D | \ \rangle = 3S_0(1)S_1(1). \tag{31}$$

Of course, it is possible to derive the last sum rule by calculating the double commutator, as was done in Eq.(15). In this case, the sum rule of the DGDR, Eq.(5), can be obtained from Eq.(26). We note also that Eq.(15) is easily obtained in employing Eq.(28).

Second, if there is a single collective state $|m = 1\rangle$ with the excitation energy ω_1, then Eq.(26) yields $\omega_{2n} = 2\omega_1$, and hence, owing to Eq.(25), we have $\omega_{mn} = m\omega_1$. More approximately, Eq.(26) implies that if $\omega_{1n} \approx \bar{\omega}$, we should have $\omega_{2n} \approx 2\bar{\omega}$, where $\bar{\omega}$ represents the mean energy of the SGDR. This fact is expressed by rewriting Eq.(26) as

$$\omega_{2n}\langle 2n | D^2 | \ \rangle = 2\sum_{n'}(\omega_{1n'} - \bar{\omega})\langle 2n | D | 1n' \rangle\langle 1n' | D | \ \rangle + 2\bar{\omega}\langle 2n | D^2 | \ \rangle. \tag{32}$$

Thus, if the width of the SGDR is narrow, the width of the DGDR is also expected to be narrow. The conclusion on other m-GDR is also the same. In other word, if the width of the SGDR is narrow, interactions satisfying $[D, [V, D]] = 0$ may not play a role to make the width of the m-GDR broader. On the one hand, the diagonal matrix element of Eq.(1) shows that there should be the same dipole strength on any state. This fact is usually called Brink's hypothesis.[8] On the other hand, its off-diagonal matrix elements provide a constraint on the distribution of the strengths.

The constraint on the width of the DGDR due to the SGDR may be seen more clearly in the following identity which is obtained from Eq.(26),

$$\sum_n (\omega_{2n} - 2\bar{\omega})^2 |\langle 2n | D^2 | \ \rangle|^2$$
$$= 4\sum_{n,n'}(\omega_{1n} - \bar{\omega})(\omega_{1n'} - \bar{\omega})\langle \ | D | 1n \rangle\langle 1n | D^2 | 1n' \rangle\langle 1n' | D | \ \rangle, \tag{33}$$

where the mean energy of the SGDR is defined by $\bar{\omega} = S_1(1)/S_0(1)$.

In the folding model, the above equation yields the relationship between the variances of the DGDR and SGDR strength functions. Eq.(19) gives

$$S_0(2) = 2S_0^2(1), \tag{34}$$

and

$$\sum_n \omega_{2n} |\langle 2n | D^2 | \ \rangle|^2 / S_0(2) = 2\bar{\omega}. \tag{35}$$

The straightforward calculation of Eq.(33), together with the above two equations and Eq.(19), provides us with the well-known result,[7),8)]

$$\sigma_2 = \sqrt{2}\sigma_1, \tag{36}$$

where σ_1 and σ_2 denote the variances of the SGDR and DGDR, respectively,

$$\sigma_1^2 = \sum_n (\omega_{1n} - \bar{\omega})^2 |\langle 1n \,|\, D \,|\ \rangle|^2 / S_0(1), \tag{37}$$

$$\sigma_2^2 = \sum_n (\omega_{2n} - 2\bar{\omega})^2 |\langle 2n \,|\, D^2 \,|\ \rangle|^2 / S_0(2). \tag{38}$$

It should be noted that the broadening by the factor $\sqrt{2}$ in Eq.(36) is not due to nuclear interactions. This is just a stochastic factor given for the two-dimensional probability distribution, when there is no correlation between the two stochastic variables. In the folding model, the stochastic variable is the excitation energies of the bosons, and the probability function is related to the transition strength function. Since the covariance in the two boson sets is zero in the folding model, the only first term of the right hand side in Eq.(19) contributes to the right hand side of Eq.(33). Eqs.(34) and (35) are also derived from a stochastic point of view. In the folding model, we have $\sigma_m = \sqrt{m}\sigma_1$.

In summary, various sum rules for multiple giant dipole states are derived by assuming, in the same way as for the Thomas-Reiche-Kuhn sum rule, that the double commutator of the dipole operator with the nuclear Hamiltonian is a constant. Nuclear interactions depending only on nucleon coordinates, exchange forces with zero-range and spin-orbit forces satisfy this assumption. It has been discussed that the sum rule for the triple giant dipole resonance can not be obtained without further assumptions.

The sum rules are derived from the diagonal matrix elements of the double commutator. We have shown that non-diagonal matrix elements also provide us with useful identities, which yield constraints on the excitation energy and transition strength of each nuclear state. By using those identities, we can discuss qualitatively the relationship between the width of the single giant dipole resonance and those of the multiple ones. If the width of the single dipole giant resonance is narrow, then that of the multiple one is expected to be also narrow. For more quantitative understanding of excited states, and effects of exchange forces with finite range or velocity-dependent forces, of course, more elaborate calculations are required.

Finally, we note that in fact, throughout this paper, we did not use the explicit form of the dipole operator, but assumed only the double commutator of the excitation operator with the Hamiltonian to be constant. Therefore, the present study may be useful not only in nuclear physics, but also in discussions of other quantum systems like atomic clusters and Bose-Einstein condensation. Moreover, collective excitations other than the dipole one may be discussed in the same way. The new identities, derived from the non-diagonal matrix elements of the double commutator, may be useful for discussions on anharmonicity of collective motions in many-body systems.

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